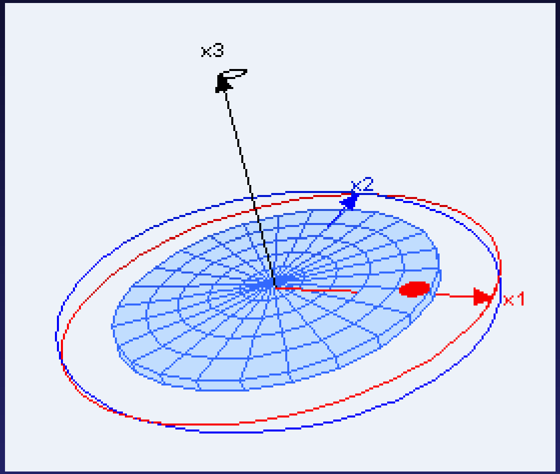
Feynman's Wobbling Plate

Why Circles?

More Detailed Explanation



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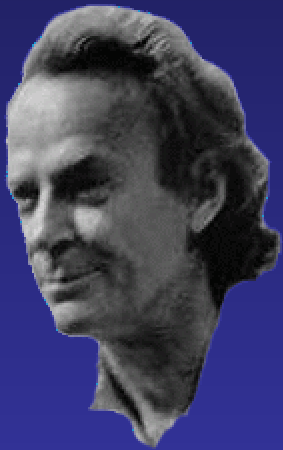
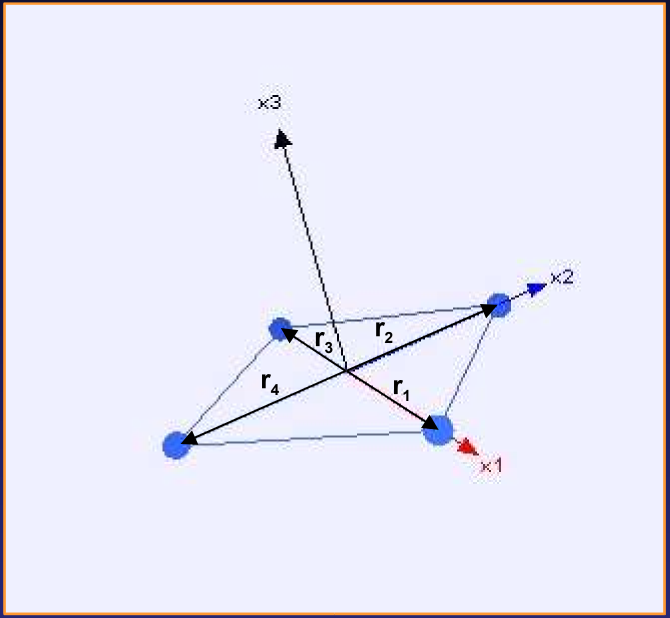


Simplification – The Elementary Plate



$$\vec{r}_3 = -\vec{r}_1$$

$$\vec{r}_2 = -\vec{r}_4$$

Plan of Attack

- Explore accelerations of the four particles of the elementary plate.
- Use symmetry to reduce to two particles.
- Decompose these accelerations to their radial and tangential components and prove that for a small magnitude of wobbling these accelerations are ***centripetal:*** they have only radial components; their *tangential components* ***vanish*.**

Plan of Attack

- Finally notice that the motion of a real plate can be understood as a result of the synchronous motion of all the "elementary" plates (groups of four particles) it can be composed of.

Balancing Accelerations

Each particle of the elementary plate feels a force

$$\vec{F}_i = m_i \ddot{\vec{r}}_i \qquad \text{where } i = 1, 2, 3, 4$$

These forces exert a zero torque on the plate

$$\vec{0} = \sum_i \vec{r}_i \times \vec{F}_i = \sum_i \vec{r}_i \times (m_i \ddot{\vec{r}}_i)$$

Balancing Accelerations

$$m(\vec{r}_1\times\ddot{\vec{r}}_1+\vec{r}_2\times\ddot{\vec{r}}_2+\vec{r}_3\times\ddot{\vec{r}}_3+\vec{r}_4\times\ddot{\vec{r}}_4)=\vec{0}$$

$$\vec{r}_3=-\vec{r}_1 \qquad \vec{r}_2=-\vec{r}_4$$

This yields something, Feynman calls *balancing accelerations*

$$\boxed{\vec{r}_1\times\ddot{\vec{r}}_1+\vec{r}_2\times\ddot{\vec{r}}_2=\vec{0}}$$

Components of Accelerations

Decompose accelerations into their radial and tangential components and substitute into the acceleration balance equation.

$$\vec{a}_1 = \vec{a}_{1\ \mathrm{rad}} + \vec{a}_{1\ \mathrm{tan}}$$

$$\vec{r}_1 \times \vec{a}_1 = \underbrace{\vec{r}_1 \times \vec{a}_{1\ \mathrm{rad}}}_{0} + \vec{r}_1 \times \vec{a}_{1\ \mathrm{tan}} = \vec{r}_1 \times \vec{a}_{1\ \mathrm{tan}}$$

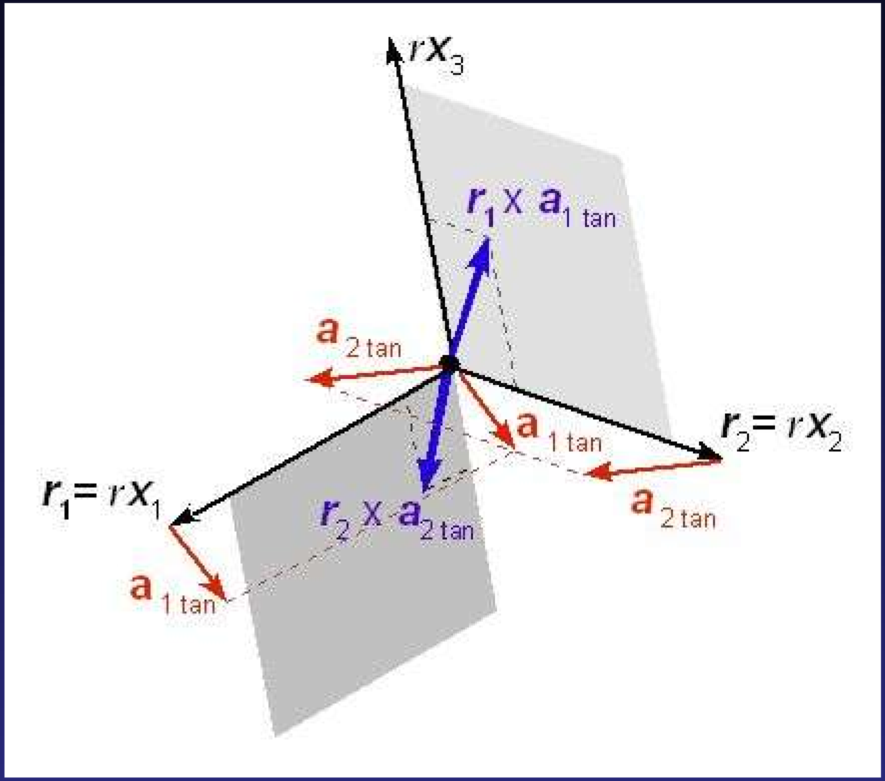
Components of Accelerations

$$\vec{r}_1 \times \vec{a}_1 + \vec{r}_2 \times \vec{a}_2 = \vec{0}$$

this becomes this

$$\boxed{\vec{r}_1 \times \vec{a}_{1\,\tan} + \vec{r}_2 \times \vec{a}_{2\,\tan} = \vec{0}}$$

Directions and Magnitudes of Accelerations

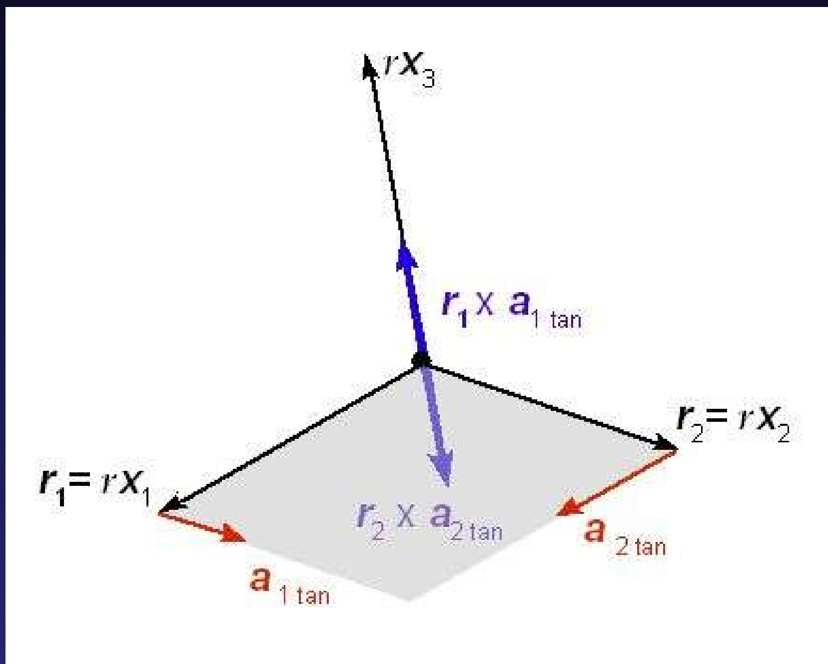


$$\vec{r}_1 \times \vec{a}_{1\,\tan} + \vec{r}_2 \times \vec{a}_{2\,\tan} = \vec{0}$$

How to satisfy this condition?

Wrong way to satisfy it ...

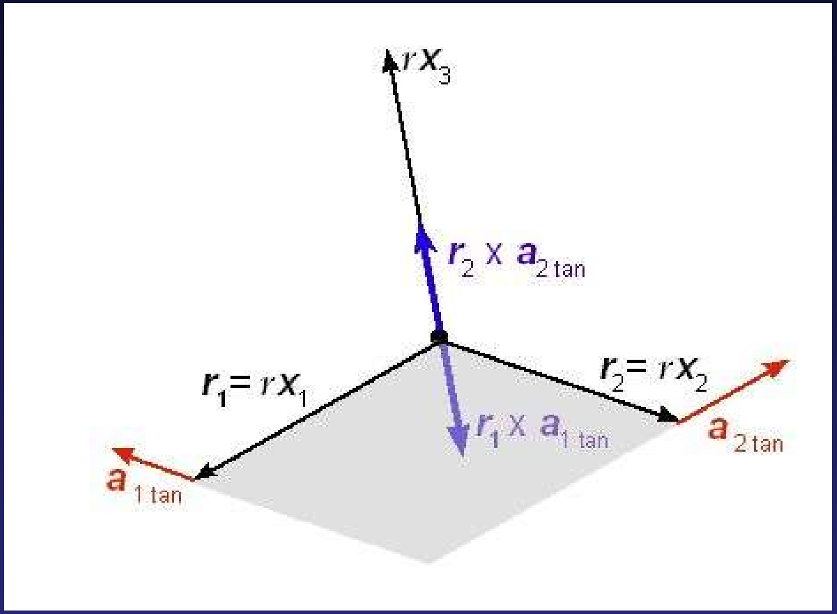
Directions and Magnitudes of Accelerations



$$\vec{r}_1 \times \vec{a}_{1\,\tan} + \vec{r}_2 \times \vec{a}_{2\,\tan} = \vec{0}$$

The first possibility!

Directions and Magnitudes of Accelerations



$$\vec{r}_1 \times \vec{a}_{1\,\tan} + \vec{r}_2 \times \vec{a}_{2\,\tan} = \vec{0}$$

Notice that in both cases $\left|\vec{a}_{1\,\tan}\right| = \left|\vec{a}_{2\,\tan}\right|$

The second possibility

The Elementary Plate is Rigid

$$\vec{r}_1 \cdot \vec{r}_1 = r^2 \xrightarrow{\quad \frac{d}{dt} \quad} \vec{r}_1 \cdot \dot{\vec{r}}_1 = 0$$

$$\vec{r}_2 \cdot \vec{r}_2 = r^2 \xrightarrow{\quad \frac{d}{dt} \quad} \vec{r}_2 \cdot \dot{\vec{r}}_2 = 0$$

$$\vec{r}_1 \cdot \vec{r}_2 = 0$$

Magnitude of Tangential Accelerations

$$\vec{r}_1 \cdot \vec{r}_2 = 0 \xrightarrow{\frac{d^2}{dt^2}} \ddot{\vec{r}}_1 \cdot \vec{r}_2 + \ddot{\vec{r}}_2 \cdot \vec{r}_1 + 2\,\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2 = 0$$

After decomposing the accelerations:

$$\underbrace{\vec{a}_{1\,\mathrm{rad}} \cdot \vec{r}_2}_{0} + \underbrace{\vec{a}_{1\,\mathrm{tan}} \cdot \vec{r}_2}_{\pm r \left|\vec{a}_{\mathrm{tan}}\right|} + \underbrace{\vec{a}_{2\,\mathrm{rad}} \cdot \vec{r}_1}_{0} + \underbrace{\vec{a}_{2\,\mathrm{tan}} \cdot \vec{r}_1}_{\pm r \left|\vec{a}_{\mathrm{tan}}\right|} + 2\,\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2 = 0$$

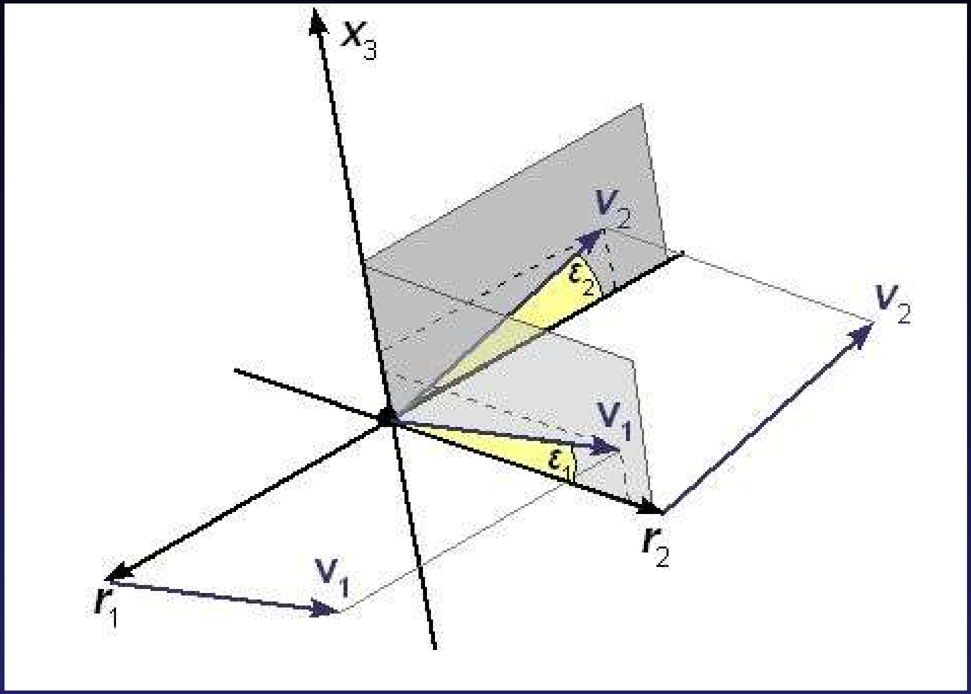
Magnitude of Tangential Accelerations

$$\pm 2\, r\left|\vec{a}_{\tan}\right| + 2\,\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2 = 0$$

Finally we obtain the expression for the magnitude of the tangential acceleration:

$$\boxed{\left|\vec{a}_{\tan}\right| = \left|\frac{\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2}{r}\right|}$$

Small Magnitude of Wobbling



$$\vec{v}_1 = \dot{\vec{r}}_1 = \left|\dot{\vec{r}}_1\right| \cos\varepsilon_1 \, \hat{x}_2 + \left|\dot{\vec{r}}_1\right| \sin\varepsilon_1 \, \hat{x}_3$$

$$\vec{v}_2 = \dot{\vec{r}}_2 = -\left|\dot{\vec{r}}_2\right| \cos\varepsilon_2 \, \hat{x}_1 + \left|\dot{\vec{r}}_2\right| \sin\varepsilon_2 \, \hat{x}_3$$

Small Magnitude of Wobbling

$$\left|\vec{a}_{\tan}\right| = \left|\frac{\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2}{r}\right|$$

$$\boxed{\left|\vec{a}_{\tan}\right| = \frac{\left|\dot{\vec{r}}_1\right|\left|\dot{\vec{r}}_2\right|}{r} \sin\varepsilon_1 \sin\varepsilon_2 \approx \frac{\left|\dot{\vec{r}}_1\right|\left|\dot{\vec{r}}_2\right|}{r} \varepsilon_1\, \varepsilon_2 \to 0}$$

Adding Elementary Plates

- The real plate can be composed of a large number of “elementary” (four-particle) plates.
- All the particles of those elementary plates have only radial accelerations and thus only radial interaction.
- Synchronous motion of all those elementary plates gives rise to the motion of the real plate.

Conclusions

- We have shown that acceleration of every particle of the plate is centripetal, and THEREFORE
- every particle of the plate will trace out a circle. QED

Thank you.

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